Powerful Ways of Thinking: Blowing Things out of (or into) Proportion

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Allow Me to Explain…

Mathematics learning, at a minimum, means learning:

- Ways of thinking that are empowered by…
- Habits of thinking, along with an ability to make sense of and use some…
- Ways of doing to do something bigger with mathematics.

Ways of Doing?

2014 – 1941

Step 1: Notice the subtraction exercise.

$$2014 - 1941$$

Step 2: Set up the problem in a template.

$$\begin{array}{r} 2014 \\ \underline{-1941} \end{array}$$

Step 3: Work from right to left. Subtract 1 from 4.

$$\begin{array}{r} 2014 \\ \underline{-1941} \\ 3 \end{array}$$

Step 4. Got it. Now move one column to the left. Subtract 4 from 1.

Step 5. Realize that I cannot subtract 4 from 1.

Step 6. Remember that I need to “borrow” from the next column to the left.

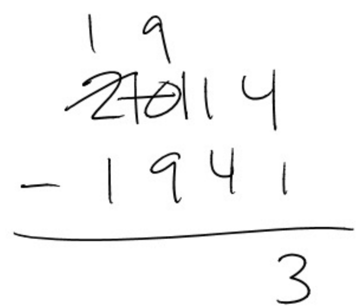
Step 7. Notice there is nothing to borrow. (Perhaps also consider how I wasn’t going to ever give it back anyway.)

Step 8. Borrow from the 2 in the far left column. Make it a 1.

Step 9. Think of the “0” in the top row as a 10. Wedge a tiny 1 between the 2 and the 0. Some might want to make it a little higher than the 0.

Step 10. Now borrow from the newly made 10. Make it a 9 by crossing out the 0 and 1 you just created during the previous step.

Step 11. Wedge a tiny 1 to the left of the original 1. I now have something that looks like this:



Step 12. Subtract 4 from the newly-made 11. Get 7.

Step 13. Subtract 9 from 9.

Step 14. Subtract 1 from 1.

Step 15. Declare the answer to be 73.

Jordan Ellenberg:

But, too often, we teach our students that “doing mathematics” means “manipulating clusters of digits according to rules presented to us by the teacher.”

That’s not math. And when we teach our students to do that, and only that, we are training them to be slow, buggy versions of Excel. What’s the point?

http://www.slate.com/blogs/how_not_to_be_wrong/2014/06/03/number_sentences_stephen_colbert_thinks_they_re_silly_they_re_not.html

Ways of Thinking (Example)

- How many years was it from December 7, 1941, to December 7, 2014?
- How did you figure it out?

One possibility:

- There are 59 years from 1941 to 2000.
- There are 14 years from 2000 to 2014.
- That means there are 59+14 = 73 years between 1941 and 2014.

Another:

Shift the whole thing by one year. The elapsed time from 1941 to 2014 is the same as 1940 to 2013. From here a counting up method is exceptionally efficient.

You have three broomsticks

The RED broomstick is three feet long

The YELLOW broomstick is four feet long

The GREEN broomstick is six feet long

3 ≠ 2

The RED broomstick is three feet long

The YELLOW broomstick is four feet long

The GREEN broomstick is six feet long

How much longer is the GREEN broomstick than the RED broomstick?

3 FEET	2 TIMES AS LONG

The RED broomstick is three feet long

The YELLOW broomstick is four feet long

The GREEN broomstick is six feet long

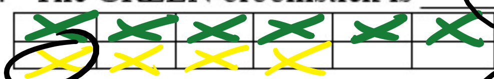
1. How much longer is the GREEN broomstick than the RED broomstick?

3 FEET	2 TIMES AS LONG

2. How much longer is the YELLOW broomstick than the RED broomstick?

1 FOOT	?

3. The GREEN broomstick is $\frac{6}{4}$ times as long as the YELLOW broomstick.

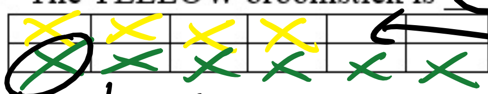


$\frac{1}{4}$ of yellow

6 copies of $\frac{1}{4}$ of yellow

$6 \cdot \frac{1}{4} = \frac{6}{4}$

The YELLOW broomstick is $\frac{4}{6}$ times as long as the GREEN broomstick.

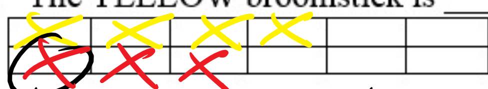


$\frac{1}{6}$ of green

4 copies of $\frac{1}{6}$ of green

$4 \cdot \frac{1}{6} = \frac{4}{6}$

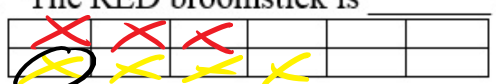
The YELLOW broomstick is _________ times as long as the RED broomstick.



$\frac{1}{3}$ of red

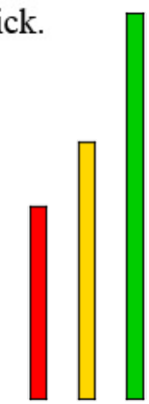
$4 \cdot \frac{1}{3} = \frac{4}{3}$

The RED broomstick is _______ times as long as the YELLOW broomstick.



$\frac{1}{4}$ of yellow

$3 \cdot \frac{1}{4} = \frac{3}{4}$



A certain stock started at a value $74. One year later it was valued at $89.54. By what percent did the stock's value increase?

$$\frac{89.54}{74}$$

HOW MANY COPIES OF 74 ARE IN 89.54? 1.21

The Willis tower (formerly the Sears tower) is 1730 feet high.
The Burj Khalifa (formerly Burj Dubai) is 2717 feet high.

a. The Burj is _______________ times as large as the Willis tower.

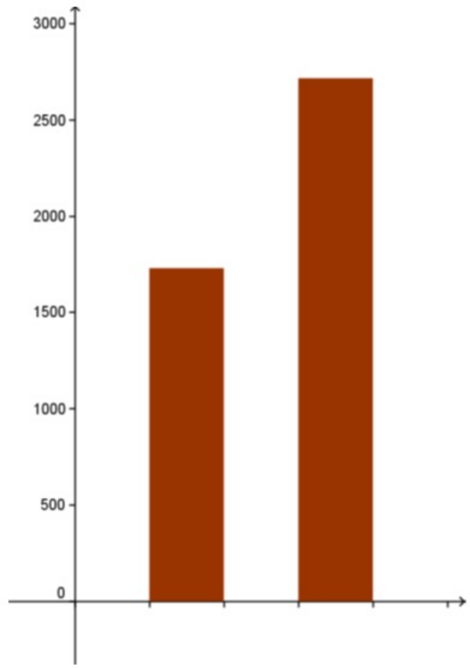
$\frac{2717}{1730} \approx 1.57$

b. The Willis tower is ______________times as large as the Burj

$\frac{1730}{2717} \approx .64$

c. The Burj is ______________ percent the size of the Willis tower.

d. The Willis tower is ______________ percent the size of the Burj.



How can you think about $\frac{a}{b} = c$?

Write two phrases that use *a*, *b*, and *c* and either “copies of” or “times as large as.”

- **There are** c **copies of** b **in** a
- a **is** c **times as large as** b

Measurement

What do we mean when we talk about “measurement”?

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Measurement

"Technically, a *measurement* is a number that indicates a comparison between the attribute of an object being measured and the same attribute of a given unit of measure."
–Van de Walle (2001)

But what is meant by "comparison"?

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Van de Walle, J. A., (2001). Elementary and middle school mathematics: Teaching developmentally, 4th edition.

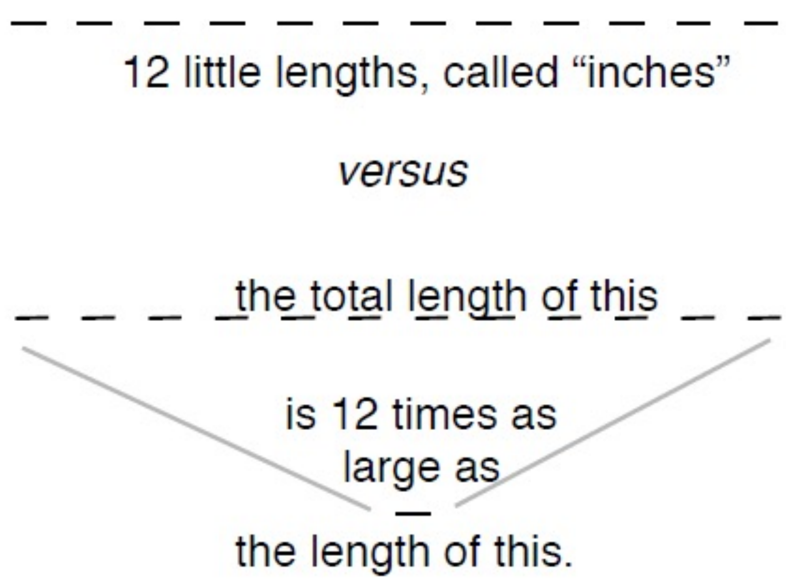

Figure 4. Measure as a number of things vs. measure as a ratio comparison.

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), Research companion to the Principles and Standards for School Mathematics (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.(pdf p. 22)

Measurement

How about this?

- Determine the attribute you want to measure
- Find something else with the same attribute. Use it as the measuring unit.
- Compare the two: *multiplicatively.*

NCTM Teaching Practices

Effective Mathematics Teaching Practices

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Meaningful mathematics discourse must center on mathematical meanings!

https://www.nctm.org/Conferences-and-Professional-Development/Principles-to-Actions-Toolkit/Resources/7-EffectiveMathematicsTeachingPractices/

4.OA

1. Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.

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Source: Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. P.29

5.NF

- 5a. Interpret multiplication as scaling (resizing), by:
- Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

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Source: Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. P.36

- "In Grades 6 and 7, rate, proportional relationships and linearity build upon this scalar extension of multiplication. Students who engage these concepts with the unextended version of multiplication (*a* groups of *b* things) will have prior knowledge that does not support the required mathematical coherences."

Source: Learning Trajectories in Mathematics: A Foundation for Standards, Curriculum, Assessment, and Instruction. Daro, et al., 2011. p.49

What terms/words come to mind when you hear the word “proportional”?

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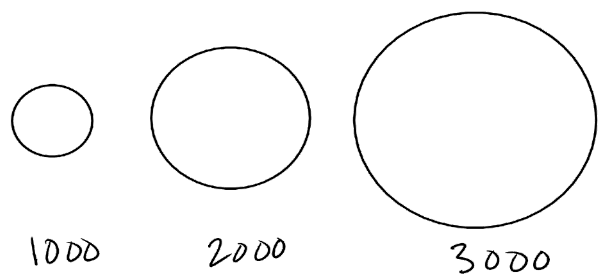
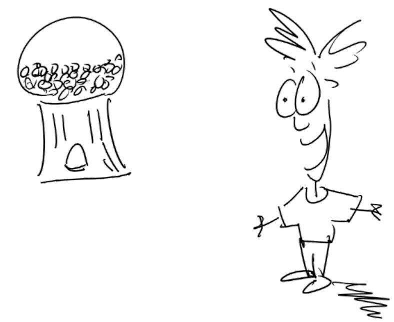
- Suppose the following graphic represents a population over a three-year period. When you look at the graphic do you think that something is out of proportion?



- If someone were to receive a fifteen-year prison sentence for stealing a gumball would you consider that to be out of proportion?



- In middle school mathematics you learned about proportional relationships.
- Are all of these the same meaning of proportionality?

Ways of Thinking (Another Example)

- What does it really mean to say something is out of proportion?
- Does your meaning extend to:

- ☐ The equation $\mathbf{y} = \mathbf{rx}$
- ☐ The equation $\frac{\mathbf{a}}{\mathbf{b}} = \frac{\mathbf{c}}{\mathbf{d}}$
- ☐ Saying that **a** is inversely proportional to **b**
- ☐ Your significant other saying you really blew that issue way out of proportion (this is technically a mathematical claim)

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“What does it mean in general to say that one quantity is proportional to another quantity? Be as precise as you can.”

Quote from Dick Stanley at http://blogs.ams.org/matheducation/2014/11/20/proportionality-confusion/

“The confusing jumble of responses here is disturbing. At the very least it points to a lack of a common understanding within the school mathematics community of this very basic and important subject.”

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Quote from Dick Stanley at http://blogs.ams.org/matheducation/2014/11/20/proportionality-confusion/

Proportional Relationships

Decluttered

Many common scenarios in the world, in everyday life, and in mathematics involve two or more quantities that we can naturally measure,

- whose measures can or do adopt a variety of possible values,
- whose measures seem related to each other, and
- the measures “scale in tandem.”

https://www.nwea.org/resource-library/map-growth-research-guidance/proportional-relationships-decluttered-white-paper-oct20-3

- Definition*: Two measurable quantities in a given scenario that can, or do, vary in value are said to be in a proportional relationship if they “scale in tandem,” that is, if the measure of one quantity in the scenario changes by a factor k, then the measure of the other quantity is sure to also change by the same factor k.

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* See Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), Research companion to the Principles and Standards for School Mathematics (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Yes or no? Why?

- Exchange rates? Say €100 = $114 …
- Walking at a constant rate? 5 miles every hour …
- Percentages? 720 people represent 60% of a population …
- Squares? Perimeter and area …
- Time to dry socks on a clothesline? 1 sock takes 20 minutes …

I can buy 27 Kewpie dolls for $15…

$$
\begin{aligned}
1.8 \text{ dolls} &\longleftrightarrow 1 \text{ dollar} \\
9 \text{ dolls} &\longleftrightarrow 5 \text{ dollars} \\
27 \text{ dolls} &\longleftrightarrow 15 \text{ dollars} \\
54 \text{ dolls} &\longleftrightarrow 30 \text{ dollars} \\
540 \text{ dolls} &\longleftrightarrow 300 \text{ dollars} \\
1{,}620 \text{ dolls} &\longleftrightarrow 900 \text{ dollars} \\
1{,}800 \text{ dolls} &\longleftrightarrow 1{,}000 \text{ dollars} \\
1.8k \text{ dolls} &\longleftrightarrow k \text{ dollars}
\end{aligned}
$$

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135% of a quantity is 156. What is the quantity?

$$135\% \longleftrightarrow \text{value } 156$$

$$1\% \longleftrightarrow \text{value } \frac{156}{135}$$

$$100\% \longleftrightarrow \text{value } \frac{15600}{135} = 115\,\frac{5}{9}$$

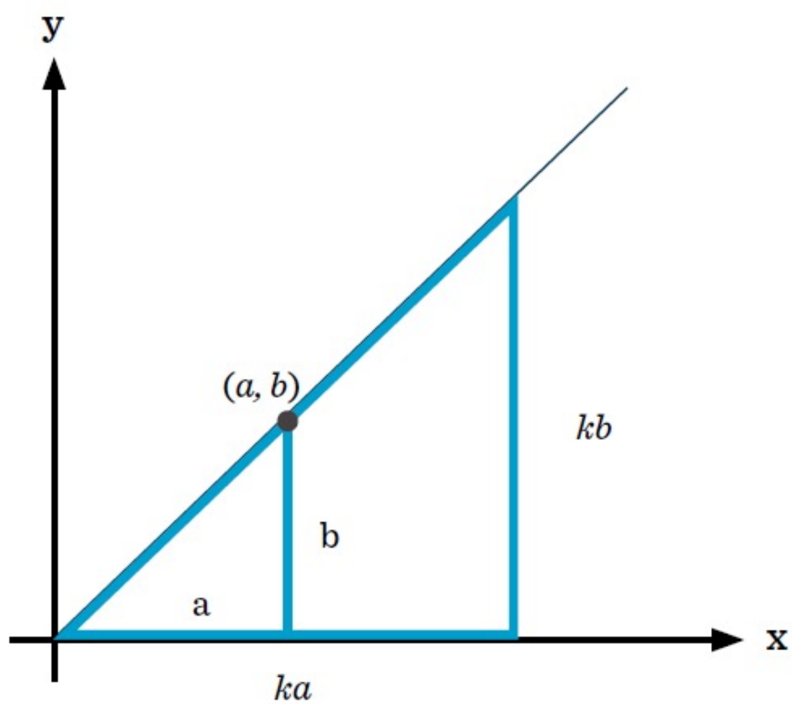
A	B
5	7
10	14
30	42
3	4.2
6	8.4

x	y
5	7
10	14
30	42
3	4.2
6	8.4
1	
x	

$$y = 1.4x$$

The data arising from any proportional relationship satisfies an equation of the form *y* = *rx* for some fixed constant *r*. And, conversely, the data that arises from any given equation of the form *y* = *rx* is in a proportional relationship.

If (a, b) is a point on a line through the origin, then similar triangles show that (ka, kb) is also on that line.



CCSS: Grade 8, 8.EE.6

6. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

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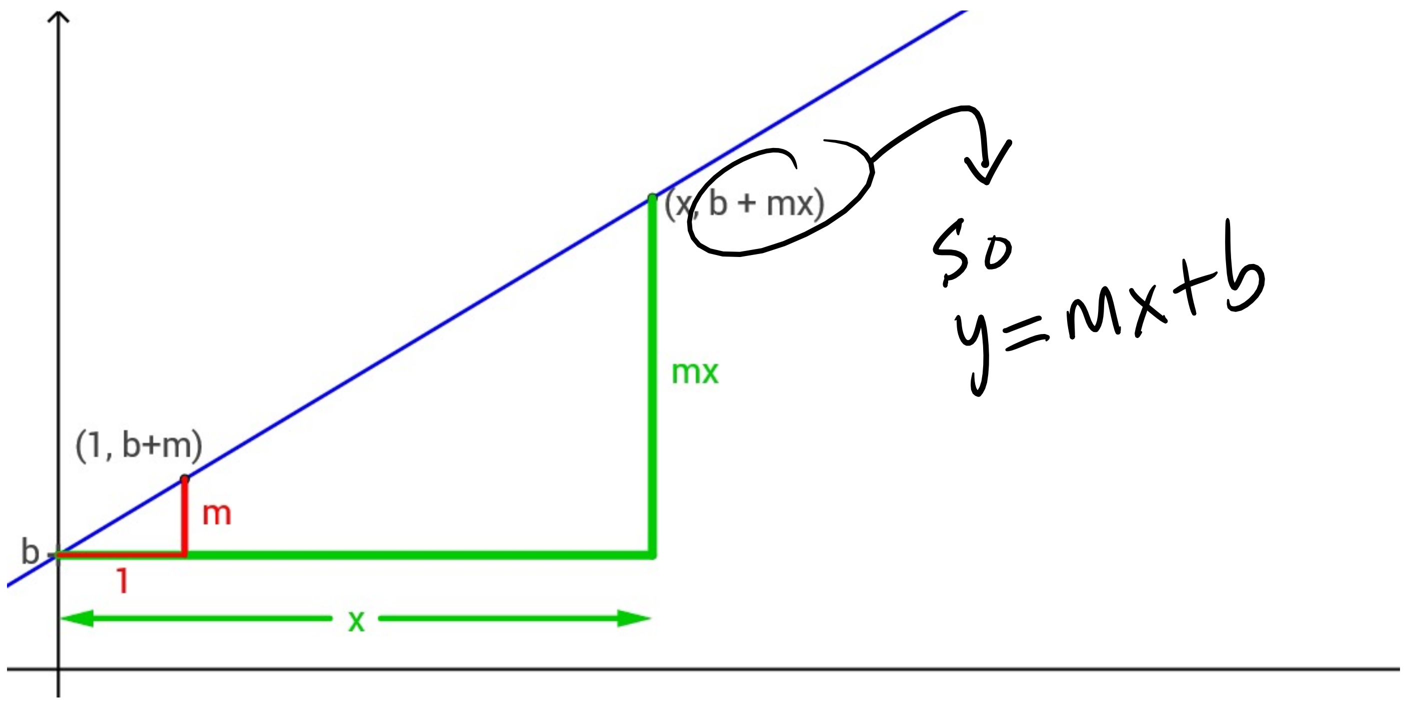
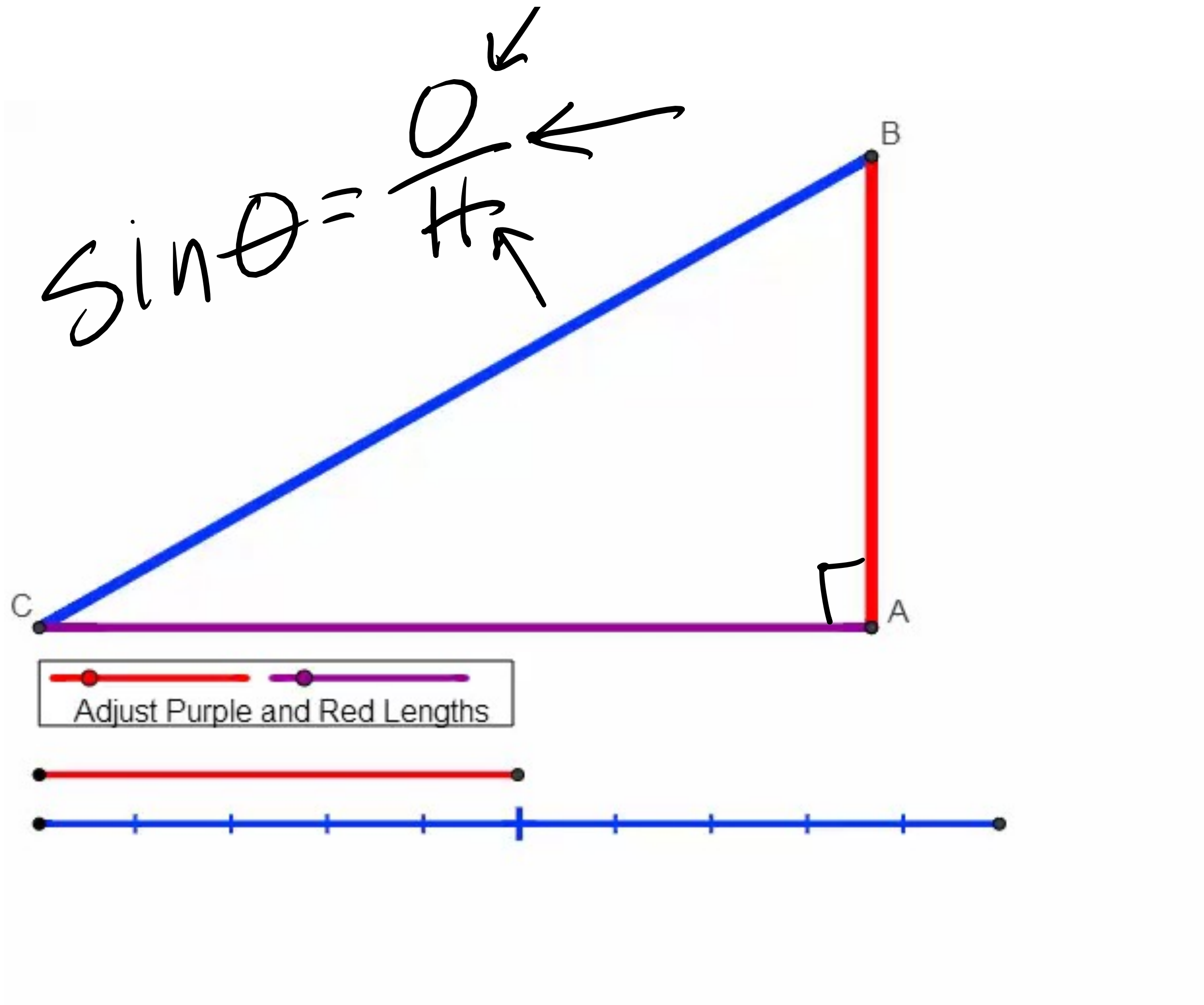
Source: Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. P.54



Adapted from: http://commoncoretools.me/wp-content/uploads/2013/07/ccss_progression_functions_2013_07_02.pdf (p.5)



http://tedcoe.com/math/geometry/similar-triangles

Re-examining Standards

- A renewed urgency to focus on mathematics that transcends "grade-level standards":
  - Ways of thinking continue to grow through ongoing grades. They need to be reinforced throughout the following years.
  - Re-packing the mathematics? It still needs to make sense – all the way down.
  - We need clear, shared meanings about (and not just around) the mathematics.

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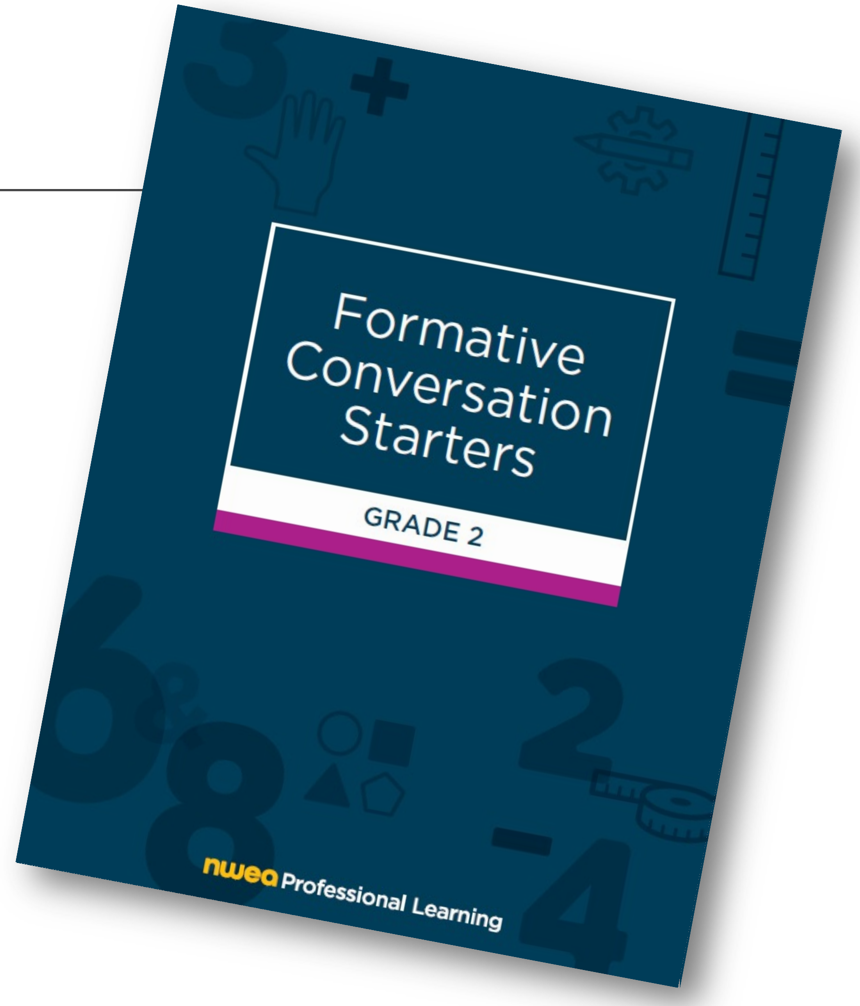
Formative Conversation Starters

- Grounded in enduring ways of thinking that transcend grade levels.
- Invite opportunities to *listen* to how students are thinking.
- Accessible, easy to use, safe, and free.



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Formative Conversation Starters
GRADE 2
nwea Professional Learning
Formative Conversation Starters
GRADE 3
nwea Professional Learning
Formative Conversation Starters
GRADE 4
nwea Professional Learning
Formative Conversation Starters
GRADE 5
nwea Professional Learning
Formative Conversation Starters
GRADE 6
nwea Professional Learning
Formative Conversation Starters
GRADE 7
nwea Professional Learning
Formative Conversation Starters
GRADE 8
nwea Professional Learning

BINS (Big Ideas to Nurture Sense-making)

- Operations
- Place Value
- Comparisons
- Measurement
- Fractions
- Formulas
- Variables
- Covarying Quantities
- Proportional Relationships
- The Equal Sign

BINS: Proportional Relationships

- Proportional relationships require two covarying quantities. Those quantities must be measurable in some way, and the measures of those quantities scale in tandem. When one quantity changes by a scale factor, the other quantity also changes by the same scale factor. For example, doubling one quantity's value results in a doubling of the other quantity's value. Students should have ways of thinking that allow them to distinguish the two varying quantities in any proportional relationship and to explain how the quantities change by the same scale factor. It is important to note that proportional relationships are not synonymous with proportions.

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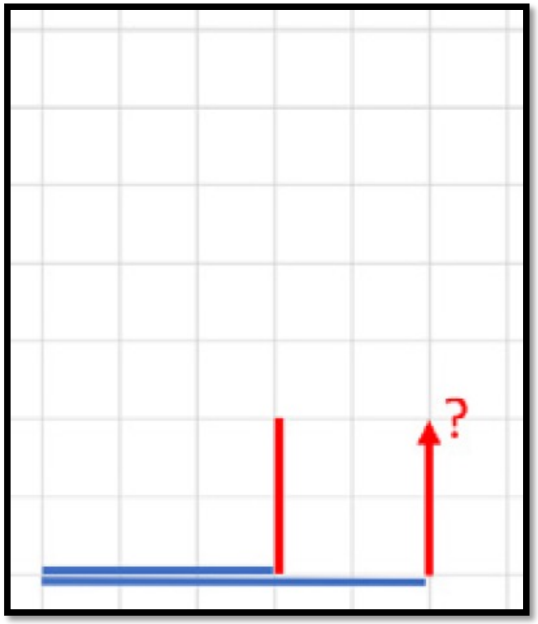
BINS: Proportional Relationships

In the image, the length of the blue line increased as shown. If the length of the red line is proportional to the length of the blue line, what is the length of the longer red line? Can you explain it without using a formula?



Are there any meaningful multiplicative comparisons in the diagram?

- What is meaningful about the number $\frac{5}{3}$ in the diagram?
- What is meaningful about the number $\frac{2}{3}$ in the diagram?
- If we imagine the blue line growing across all possible lengths, what happens with the red line? Are you sure?
- Can you write an equation that shows the relationship between the lengths of the blue and red lines?

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Practice

Sue and Julie were running around a track equally fast. Sue started before Julie.

When Sue had run 9 laps, Julie had run 3 laps.

How far had Sue run when Julie had run 15 laps?

Cramer, K. & Post, T. (1993, February). Making connections: A Case for Proportionality. *Arithmetic Teacher,* 60(6), 342-346. Hat tip to Scott Adamson.

Mathematics learning, at a minimum, means learning:

- Ways of thinking that are empowered by…
- Habits of thinking, along with an ability to make sense of and use some…
- Ways of doing to do something bigger with mathematics.

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Do your students think:

- Math is a collection of disconnected facts and algorithms
  - **Math is coherent and founded on relationships of ideas.**
- Math is idiosyncratic and situation specific
  - **Math is logical and systematic**
- Math is about quick answer-getting
  - **Math is about authentic problem solving**
- Math is only important for mathematicians
  - **Math is relevant to everyone**
- Math is to be memorized as given
  - **Math is about constructing knowledge**

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Heavily Inspired by the taxonomy of the “Views About Mathematics Survey.” For more information see: Carlson, M. P. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success. *Educational Studies in Mathematics*, *40*(3), 237-258. Only one bullet on this list is *not* from Carlson.

Do your students think:

- Math is for the talented few
  - **Math is available to anyone willing to make the effort.**
- My effort has no impact on my ability to learn math: I am not a math person.
  - **My achievement depends on my persistence.**

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Thank You

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